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2005 J. Phys. A: Math. Gen. 38 L171

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LETTER TO THE EDITOR

Physical realization of \mathcal{PT} -symmetric potential scattering in a planar slab waveguide

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Received 7 October 2004, in final form 30 December 2004

Published 16 February 2005

Online at stacks.iop.org/JPhysA/38/L171

Abstract

A physical realization of scattering by \mathcal{PT} -symmetric potentials is provided: we show that the Maxwell equations, for an electromagnetic wave travelling along a planar slab waveguide filled with gain and absorbing media in contiguous regions, can be approximated in a parameter range by a Schrödinger equation with a \mathcal{PT} -symmetric scattering potential.

PACS number: 03.65.Nk

\mathcal{PT} -symmetric Hamiltonians remain invariant under the combination of parity and time-reversal symmetry operations. They have attracted considerable attention in diverse areas such as quantum field theory [1], solid state physics [2, 3] and population biology [4]. One of their most important and striking properties is that the discrete eigenvalues are real if the eigenstates are also \mathcal{PT} -invariant or appear in conjugate pairs otherwise. Most of the work on \mathcal{PT} -invariance has dealt with discrete Hamiltonians and much less attention has been paid to scattering systems at a general or fundamental level. A few studies of the scattering by these potentials refer to specific models [5, 6] or examine transparent \mathcal{PT} -symmetric potentials [7, 8]. Some generic results, restricted to real momentum and local potentials, have been discussed by Deb *et al* [9], whereas a general formal scattering theory for one-dimensional \mathcal{PT} -symmetric potentials is provided in a recent review about complex potentials [10].

As in [10], we shall assume that the Hamiltonian of the non-relativistic particle of mass m can be written as the sum of the kinetic energy operator corresponding to the ‘free-motion’ evolution, H_0 , and the potential operator, V ,

$$H = H_0 + V. \quad (1)$$

V may be generically non-local. Consider the combined action of the anti-unitary time reversal operator \mathcal{T} ($\mathcal{T}c|x\rangle = c^*|x\rangle$) and the unitary parity operator \mathcal{P} ($\mathcal{P}c|x\rangle = c|-x\rangle$), where c is an arbitrary complex constant. ‘ \mathcal{PT} -invariant’ Hamiltonians [14] remain unchanged by this transformation,

$$[\mathcal{PT}, H] = 0. \quad (2)$$

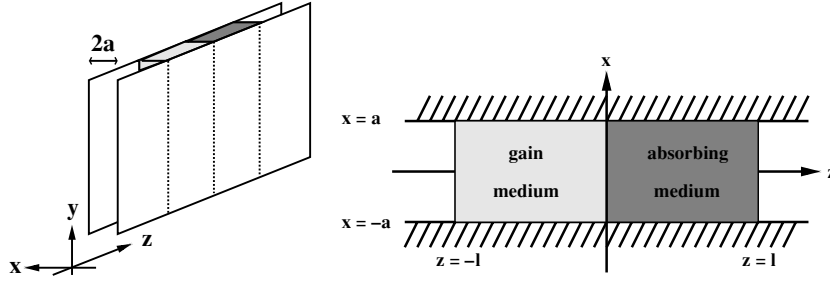


Figure 1. Schemes of the planar slab waveguide: three dimensional (left) and in the xz -plane (right).

Since the kinetic energy operator H_0 is \mathcal{PT} -invariant, this implies (note the erratum in [10])

$$\langle x|V|x'\rangle = \langle -x'|V^\dagger|-x\rangle = \langle -x|V|-x'\rangle^*. \quad (3)$$

In the particular case of local interactions, i.e., when $\langle x|V|x'\rangle = \delta(x-x')V(x)$, \mathcal{PT} -symmetry implies $V(x) = V(-x)^*$ and $V^\dagger = \mathcal{P}V\mathcal{P}$; the real part of $V(x)$ must be symmetric and the imaginary part antisymmetric. The consequences of \mathcal{PT} -symmetry in the scattering amplitudes (their structure, pole configuration and relations in the momentum complex plane) have been examined in [10].

One important open question is the possible physical meaning of non-trivial \mathcal{PT} -symmetric local interactions, i.e., non-Hermitian ones with a non-vanishing imaginary part. So far, the existing physical realizations of \mathcal{PT} -symmetry involve some non-locality, e.g., velocity-dependent potentials [3, 4]. In this letter, we show that the scattering of electromagnetic waves provides a physical realization of a local \mathcal{PT} -symmetric interaction since absorption and gain processes may be implemented at different spatial regions with appropriately chosen media.

Let us consider two ideal metallic slabs for $x < -a$ and $x > a$ forming a ‘planar waveguide’ in z and y as depicted in figure 1. In a realistic setting the waveguide must of course also have a finite width along the y -direction, but for a sufficiently large width the results of the finite width case may be arbitrarily close to those for the infinite one. In the following, we shall only consider the space between the metallic slabs, $|x| \leq a$. The waveguide is empty for $|z| > l$, with relative permittivity $\epsilon = 1$; there is also a gain medium region $-l < z < 0$ and an absorbing region $0 < z < l$. Both regions are filled with the same atomic gas, but in the gain region a laser shining in y -direction pumps the resonant atoms to their excited state to produce a population inversion [11]. The relative permittivity, assuming a simple Lorentz model, can thus be written as

$$\epsilon(z, \omega) = 1 - \xi(z) \frac{\omega_p^2}{\omega^2 - \omega_0^2 + 2i\delta\omega} \quad (4)$$

where

$$\xi(z) = \begin{cases} -1 & \text{gain medium} & -l < z < 0 \\ 1 & \text{absorbing medium} & 0 < z < l \\ 0 & \text{vacuum} & |z| > l, \end{cases} \quad (5)$$

ω_p is the plasma frequency, ω_0 is the resonance frequency and δ is a damping constant.

The Maxwell equations are

$$\begin{aligned} \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} &= 0 & \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \epsilon \left(z, i \frac{\partial}{\partial t} \right) \vec{E} &= 0 \\ \nabla \cdot \vec{E} &= 0 & \nabla \cdot \vec{B} &= 0 \end{aligned} \quad (6)$$

c being the speed of light in vacuum. We assume now the following form of an electromagnetic wave travelling in the z -direction

$$\begin{aligned} \vec{E}(x, y, z, t) &= \int d\omega \begin{pmatrix} 0 \\ -i\omega \hat{B}_x(x) \phi(z, \omega) \\ 0 \end{pmatrix} e^{-i\omega t} \\ \vec{B}(x, y, z, t) &= \int d\omega \begin{pmatrix} \hat{B}_x(x) \frac{\partial \phi}{\partial z}(z, \omega) \\ 0 \\ -\frac{\partial \hat{B}_x}{\partial x}(x) \phi(z, \omega) \end{pmatrix} e^{-i\omega t} \end{aligned} \quad (7)$$

where the physical fields are given by the real part of \vec{E} and \vec{B} . Inserting the ansatz (7) into the Maxwell equations (6), we get by separation of variables

$$\left(c^2 \frac{\partial^2}{\partial x^2} + \omega_c^2 \right) \hat{B}_x(x) = 0 \quad (8)$$

and

$$\left(k^2(z, \omega) + \frac{\partial^2}{\partial z^2} \right) \phi(z, \omega) = 0, \quad (9)$$

ω_c being the cut-off frequency and $k^2 = \frac{1}{c^2} (\omega^2 \epsilon(z, \omega) - \omega_c^2)$. The boundary and matching conditions following from classical electrodynamics [15] are $\hat{B}_x(-a) = \hat{B}_x(a) = 0$. A solution of (8) fulfilling the boundary conditions is $\hat{B}_x(x) = \cos(\pi x/2a)$ which results in $\omega_c = c\pi/2a$. Moreover, $\phi(z, \omega)$ and $\frac{\partial \phi}{\partial z}(z, \omega)$ must be continuous for all z and all ω . From the fact that $\phi(z, \omega)$ is a solution of (9) it follows that $\frac{\partial^2 \phi}{\partial z^2}(z, \omega)$ is continuous for all ω and all $z \notin \{-l, 0, l\}$ ($k^2(z, \omega)$ is non-continuous at $z = -l, 0, l$). We choose the waveguide width $2a$, so that the cut-off frequency coincides with the resonance frequency, $\omega_c = \omega_0$, and consider waves with frequencies near the cut-off, $\omega = \omega_c + \Delta\omega$. It follows that

$$\begin{aligned} k^2(z, \omega_c + \Delta\omega) &= 2 \frac{\Delta\omega\omega_c}{c^2} (1 + \Delta\omega/2\omega_c) \\ &\quad - \xi(z) \frac{\omega_p^2}{c^2} \frac{1 + 2 \frac{\Delta\omega}{\omega_c} (1 + \Delta\omega/2\omega_c)}{2 \frac{\Delta\omega}{\omega_c} (1 + \Delta\omega/2\omega_c) + 2i(1 + \Delta\omega/\omega_c)\delta/\omega_c} \\ &= i\xi(z) \frac{\omega_c\omega_p^2}{2c^2\delta} + \frac{\Delta\omega}{\omega_c} \frac{2\omega_c^2}{c^2} \left(1 - \xi(z) \frac{\omega_p^2}{4\delta^2} + i\xi(z) \frac{\omega_p^2}{4\delta\omega_c} \right) + \mathcal{O} \left(\frac{\Delta\omega}{\omega_c} \right)^2 \end{aligned} \quad (10)$$

and so, from (9),

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \phi(z, \omega_c + \Delta\omega) &= -k^2(z, \omega_c + \Delta\omega) \phi(z, \omega_c + \Delta\omega) \\ &= - \left(i\xi(z) \frac{\omega_c\omega_p^2}{2c^2\delta} + 2\Delta\omega \frac{\omega_c}{c^2} \left(1 - \xi(z) \frac{\omega_p^2}{4\delta^2} + i\xi(z) \frac{\omega_p^2}{4\delta\omega_c} \right) \right. \\ &\quad \left. + \mathcal{O} \left(\frac{\Delta\omega}{\omega_c} \right)^2 \right) \phi(z, \omega_c + \Delta\omega). \end{aligned} \quad (11)$$

We assume that $\omega_p^2/\delta \ll \delta, \omega_c$. For $\Delta\omega \ll \delta, \omega_c$, we get

$$k^2(z, \omega_c + \Delta\omega) \approx \frac{2\omega_c}{c^2} \Delta\omega + i\xi(z) \frac{\omega_c \omega_p^2}{2c^2 \delta} =: \tilde{k}_{\omega_c}^2(z, \Delta\omega)$$

and

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \phi(z, \omega_c + \Delta\omega) &\approx - \left(i\xi(z) \frac{\omega_c \omega_p^2}{2c^2 \delta} + 2\Delta\omega \frac{\omega_c}{c^2} \right) \phi(z, \omega_c + \Delta\omega) \\ \Rightarrow \Delta\omega \phi(z, \omega_c + \Delta\omega) &\approx - \frac{c^2}{2\omega_c} \frac{\partial^2}{\partial z^2} \phi(z, \omega_c + \Delta\omega) - i\xi(z) \frac{\omega_p^2}{4\delta} \phi(z, \omega_c + \Delta\omega). \end{aligned} \quad (12)$$

For $\psi(z, t) := \int d\Delta\omega \phi(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t}$, we get

$$\begin{aligned} \frac{\partial \psi}{\partial t}(z, t) &= -i \int d\Delta\omega \Delta\omega \phi(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t} \quad \forall z, t \\ \frac{\partial^2 \psi}{\partial z^2}(z, t) &= \int d\Delta\omega \frac{\partial^2 \phi}{\partial z^2}(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t} \quad \forall z, t \text{ with } z \notin \{-l, 0, l\}. \end{aligned}$$

Note that $\psi(z, t)$ is related directly to the magnetic field component $B_z(x, y, z, t) = -\frac{\partial \hat{B}_x}{\partial x}(x) e^{-i\omega_c t} \psi(z, t)$.

We assume that $\phi(z, \omega) = 0$ for $|\omega - \omega_c| \geq \Omega$ for some $\Omega \ll \delta, \omega_c$. One finds, using (12),

$$\begin{aligned} i \frac{\partial \psi}{\partial t}(z, t) &= \int d\Delta\omega (\Delta\omega) \phi(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t} \\ &\stackrel{(12)}{\approx} \int d\Delta\omega \left(-\frac{c^2}{2\omega_c} \frac{\partial^2}{\partial z^2} \phi(z, \omega_c + \Delta\omega) - i\xi(z) \frac{\omega_p^2}{4\delta} \phi(z, \omega_c + \Delta\omega) \right) e^{-i\Delta\omega t} \\ &= -\frac{c^2}{2\omega_c} \int d\Delta\omega \frac{\partial^2}{\partial z^2} \phi(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t} \\ &\quad - i\xi(z) \frac{\omega_p^2}{4\delta} \int d\Delta\omega \phi(z, \omega_c + \Delta\omega) e^{-i\Delta\omega t} \\ &= -\frac{c^2}{2\omega_c} \frac{\partial^2}{\partial z^2} \psi(z, t) - i\xi(z) \frac{\omega_p^2}{4\delta} \psi(z, t). \end{aligned} \quad (13)$$

To make this equation formally equal to a Schrödinger equation we multiply (13) with \hbar and introduce an auxiliary ‘mass’ $m = \hbar\omega_c/c^2$. Then, we can write (13) in the following form:

$$i\hbar \frac{\partial \psi}{\partial t}(z, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \psi + V(z)\psi \quad (14)$$

where we have defined an ‘effective potential’ $V(z) = -i\xi(z) \frac{\omega_p^2 \hbar}{4\delta}$ which is \mathcal{PT} -symmetric. There is in fact a physical motivation for this mass: quantum mechanically we can view the classical electromagnetic field as a mean field consisting of many photons, and we are considering here that all photons have approximately the frequency ω_c . A single photon with frequency ω_c has the energy $\hbar\omega_c$ and the momentum $p_c = \hbar\omega_c/c$. A photon has rest mass zero but nevertheless we can view the momentum as the product of velocity and a velocity-dependent ‘mass’ m . In the case of a single photon we get, precisely, $m = p_c/c = \hbar\omega_c/c^2$.

Conversely, let $\psi(z, t)$ be a continuously differentiable solution of (14) and we define $\phi(z, \omega) := \frac{1}{2\pi} \int dt \psi(z, t) \exp(i(\omega - \omega_c)t)$. If $\phi(z, \omega) < \infty$ for all z, ω , and $\phi(z, \omega) = 0$ for

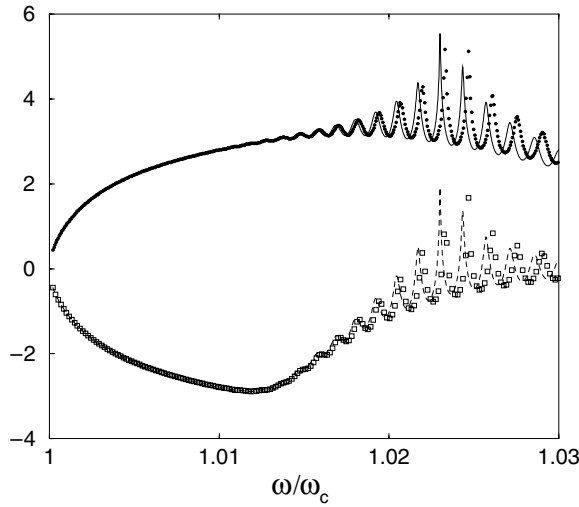


Figure 2. \log_{10} of $|T^l|^2 + |R^l|^2$ (solid line) and of $|T^r|^2 + |R^r|^2$ (dashed line) for the exact solution (9) and for the stationary solution of the approximation (14) (filled and unfilled symbols) versus ω/ω_c . $2a = 0.124 \mu\text{m}$, $\hbar\omega_c = \hbar\omega_0 = 5 \text{ eV}$, $\hbar\omega_p = 0.2 \text{ eV}$, $2\hbar\delta = 2.5 \text{ eV}$, $l = 19.7 \mu\text{m}$.

$|\omega - \omega_c| \geq \Omega$ for some $\Omega \ll \delta$, ω_c , then

$$\begin{aligned}
 \frac{\partial^2 \phi}{\partial z^2}(z, \omega) &= \frac{1}{2\pi} \int dt \frac{\partial^2}{\partial z^2} \psi(z, t) \exp(i(\omega - \omega_c)t) \\
 &= \frac{1}{2\pi} \int dt \left(-i \frac{2m}{\hbar} \frac{\partial}{\partial t} \psi(z, t) + \frac{2m}{\hbar^2} V(z) \psi(z, t) \right) \exp(i(\omega - \omega_c)t) \\
 &\stackrel{(*)}{=} \frac{1}{2\pi} \int dt \left(i \frac{2m}{\hbar} \psi(z, t) \frac{\partial}{\partial t} \exp(i(\omega - \omega_c)t) + \frac{2m}{\hbar^2} V(z) \psi(z, t) \exp(i(\omega - \omega_c)t) \right) \\
 &= -\frac{2m}{\hbar} (\omega - \omega_c) \phi(z, \omega) + \frac{2m}{\hbar^2} V(z) \phi(z, \omega) \\
 &= -\left(\frac{2\omega_c}{c^2} (\omega - \omega_c) + i \frac{\omega_c \omega_p^2}{2c^2 \delta} \xi(z) \right) \phi(z, \omega) \\
 &= -\tilde{k}_{\omega_c}^2(z, \omega - \omega_c) \phi(z, \omega) \approx -k^2(z, \omega) \phi(z, \omega)
 \end{aligned} \tag{15}$$

where we have used partial integration at (*). Note that a necessary condition for $\phi(z, \omega)$ being always finite is that $\psi(z, \pm\infty) = 0$ so there are no ‘boundary’ terms in the partial integration. Therefore, it follows from (15) that $\phi(z, \omega)$ is an approximate solution of (9) within the stated parameter regime.

We have checked numerically the validity of the approximation assuming a resonance frequency in the ultraviolet region, see figure 2. T^l (T^r) is defined as the transmission amplitude and R^l (R^r) as the reflection amplitude for incidence from the left (right) [10] of a harmonic plane wave. Note that for a real potential $V(z)$, from the unitarity of the scattering matrix, $|T^l|^2 + |R^l|^2 = |T^r|^2 + |R^r|^2 = 1$, but otherwise these equalities do not hold in general. Figure 2 shows the sums $|T^{l,r}|^2 + |R^{l,r}|^2$ calculated exactly, with the Maxwell equations, and approximately, i.e., with the effective Schrödinger equation (14). The approximation is very good for $1 \leq \frac{\omega}{\omega_c} < 1.02$, and is still in qualitative agreement with the exact result above that

value. Note the dominance of gain/absorption for low-energy scattering from the left/right, corresponding to the region found first by the wave.

Summarizing, we have shown that the equation for an electromagnetic wave travelling in a planar slab waveguide can be approximated in a parameter region by an effective Schrödinger equation with a scattering, localized, \mathcal{PT} -symmetric potential. This provides a physical realization of scattering off \mathcal{PT} -symmetric local potentials and opens a way for the experimental implementation and testing of theoretical results.

Acknowledgments

We acknowledge A Steinberg for useful comments. This work has been supported by Ministerio de Ciencia y Tecnología, FEDER (BFM2000-0816-C03-03, BFM2003-01003) and UPV-EHU (00039.310-13507/2001). AR acknowledges support of the German Academic Exchange Service (DAAD) and Ministerio de Educación y Ciencia.

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